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# On the computational aspects of affine moment invariants for gray-scale images

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**Abstract**

Affine moment invariants are widely used in pattern recognition and computer vision. The affine transform usually decomposed into a set of geometric transforms. The computation of these transforms relies on their relations with the geometric moments. Approximate computation of geometric moments produced a very large dynamic range that results in numerical instabilities. In this work, an exact method is used for the computation of affine moment invariants for gray level images, where the proposed method completely removes the approximation errors. Numerical experiments are performed to test the invariance of symmetric as well as asymmetric images subject to affine transformations. A comparison with the approximation method is made. The obtained results clearly explained the efficiency of the proposed method.


*Keywords:* Affine moment invariants; Exact computation; Symmetric images

## 1. Introduction

Affine moment invariants are useful features of an image as they are invariant to general linear transformations of the image. Reiss [1] and Flusser and Suk [2] independently introduced affine moment invariants and proved their applicability in simple recognition tasks. Their affine transforms were decomposed into translation, anisotropic scaling and two skews.

Rothe and his co-authors [3] proposed the concept of affine normalization. In their work, two different affine decompositions were used. The first called XSR consists of two skews, anisotropic scaling and rotation. The second is the XYS and consists of two skews and anisotropic scaling. Zhang et al. [4] performed a study of these affine decompositions and pointed out that both transformations led to some ambiguities. Pei and Lin [5] presented a method for affine normalization. Their method dealt only with asymmetrical objects. Shen and Ip [6] used the generalized complex moments in polar coordinates in recognition of symmetrical objects. Yang and Cohen [7] introduced the concept of cross-weighted moments to recover the affine transformation parameters. This method is very consuming where the complexity of computing a crossed-weighted moment of

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$M \times N$ digital image will be $O((M \times N)^2)$. Recently, Suk and Flusser [8] proposed a method for affine normalization for symmetric objects. Their method used geometric transformations as well as the complex moments.

As we see, all of the aforementioned methods used either geometric and/or complex moment in Cartesian or polar coordinates. The computation of complex moments and the affine moment invariants in polar coordinates produced two kinds of errors. The first is the numerical error which is the direct result of approximation process. Second is the geometric error that is a result of circle to square mapping. Unfortunately, all methods listed above are suffering from at least one of these errors which degraded the accuracy of the computed values. Recently, Hosny [9] proposed a new method for exact and fast computation of geometric moments, where the numerical error is completely removed. Complex moments could be computed exactly as a combination of exact geometric moments. Consequently, affine moment invariants are computed exactly.

This paper proposes a new method for accurate computation of affine moment invariants in case of symmetric as well as asymmetric gray level images and objects. A set of two-dimensional geometric moments are computed exactly by using a mathematical integration of the monomial polynomials, then complex moments and affine moments invariants are exactly calculated based on the algebraic relations with geometric moments. A fast algorithm is applied for computation complexity reduction. Experimental results clearly show the efficiency of this proposed method.

The rest of the paper is organized as follows: In Section 2, an overview of the exact and fast algorithm for geometric moment's computations is presented. Affine transformations and algebraic relations that produced affine moment invariants are described in Section 3. Section 4 is devoted to numerical experiments. Conclusion and concluding remarks are presented in Section 5.

## 2. Exact computation of geometric moments

Regular or geometric moments of order $(p+q)$ for image intensity function $f(x,y)$ are defined as

$$m_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^q f(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{1}$$

with $p,q \geqslant 0$. A digital image of size $M \times N$ is an array of pixels. Centers of these pixels are the points $(x_i, y_j)$, where the image intensity function is defined only for this discrete set of points $(x_i, y_j) \in [0, M-1] \times [0, N-1]$. $\Delta x_i = x_{i+1} - x_i$, $\Delta y_j = y_{j+1} - y_j$ are sampling intervals in the $x$-and $y$-directions, respectively. In the literature of digital image processing, the intervals $\Delta x_i$ and $\Delta y_j$ are fixed at constant values $\Delta x_i = 1$, and $\Delta y_j = 1$, respectively. Therefore, the set of points $(x_i, y_j)$ will be defined as follows:

$$x_i = \left(i - \frac{1}{2}\right)\Delta x, \tag{2.1}$$

$$y_j = \left(j - \frac{1}{2}\right)\Delta y, \tag{2.2}$$

with $i = 1, 2, 3, \ldots, M$ and $j = 1, 2, 3, \ldots, N$. For the discrete-space version of the image, Eq. (1) is usually approximated as

$$\tilde{M}_{pq} = \sum_{i=1}^{M}\sum_{j=1}^{N} x_i^p y_j^q f(x_i, y_j)\Delta x \Delta y. \tag{3}$$

Eq. (3) is the so-called direct method for geometric moment's computations, which is the approximated version using zeroth-order approximation (ZOA). Eq. (3) is not a very accurate approximation of Eq. (1). Similar to our previous work [9], the set of geometric moments can be computed exactly by

$$\hat{M}_{pq} = \sum_{i=1}^{M}\sum_{j=1}^{N} I_p(i) I_q(j) f(x_i, y_j), \tag{4}$$

where

$$I_p(i) = \frac{1}{p+1}[U_{i+1}^{p+1} - U_i^{p+1}], \tag{5.1}$$

$$I_q(j) = \frac{1}{q+1}[V_{j+1}^{q+1} - V_j^{q+1}], \tag{5.2}$$

and

$$U_{i+1} = x_i + \frac{\Delta x_i}{2}, \tag{6.1}$$

$$U_i = x_i - \frac{\Delta x_i}{2}, \tag{6.2}$$

$$V_{j+1} = y_j + \frac{\Delta y_j}{2}, \tag{6.3}$$

$$V_j = y_j - \frac{\Delta y_j}{2}. \tag{6.4}$$

## 3. Affine moment invariants

Moments are one of the parameters that describe the image or object of interest. Moment invariants are moments which do not change under a group of transformations. Image normalization means bringing the image to a position in which the effect of transformation is eliminated. Affine transformation is represented by the following matrix form:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{7}$$

To achieve normalization, affine transformation decomposed into a group of simple one-parameter transforms [3,8]. This group consists of translation, uniform scaling, first rotation, stretching, and second rotation

$$x' = x - x_0, \quad y' = y - y_0, \tag{8.1}$$

$$x' = \alpha x, \quad y' = \alpha y, \tag{8.2}$$

$$x' = x\cos\theta - y\sin\theta, \quad y' = x\sin\theta + y\cos\theta, \tag{8.3}$$

$$x' = \delta x, \quad y' = \frac{1}{\delta}y, \tag{8.4}$$

$$x' = x\cos\phi - y\sin\phi, \quad y' = x\sin\phi + y\cos\phi, \tag{8.5}$$

where $(x_0, y_0)$ is the centroid; $\alpha, \delta > 0$; $\theta, \phi$ are the rotation angles. The image function is invariant under the group of transformations (8) if and only if it is invariants under the general affine transformation (7).

### 3.1. Normalization to translation

Normalization to translation is achieved by shifting the image so that the image centroid $(\bar{x}, \bar{y})$ coincides with the origin of the coordinate system. The centroid of the image is

$$\bar{x} = \frac{m_{10}}{m_{00}}, \quad \bar{y} = \frac{m_{01}}{m_{00}}. \tag{9}$$

The central moments

$$\mu_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\bar{x})^p(y-\bar{y})^q f(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{10}$$

are translation invariants. By using the binomial theorem, central moments are expressed as a linear combination of regular moments of the same order or less. Therefore, the last equation can be written as

$$\mu_{pq} = \sum_{k=0}^{p} \sum_{j=0}^{q} \binom{p}{k} \binom{q}{j} (-\bar{x})^{p-k} (-\bar{y})^{q-j} m_{kj}. \tag{11}$$

According to Eq. (11), exact computation of geometric moments results in exact values for central moments. The time-consuming direct computations of factorial terms are avoided by using the following recurrence relations:

$$D(p,k) = \frac{p}{p-k} D(p-1,k), \tag{12.1}$$

$$D(p,k) = \frac{1}{k(p-k)} D(p,k-1), \tag{12.2}$$

$$D(0,0) = 1, \quad \text{and} \quad D(p,0) = 1, \tag{12.3}$$

where matrix D is created and stored for future use.

### 3.2. Normalization to scaling

Assume that $\alpha$ is a scaling factor in $x$- and $y$-directions, respectively. Any positive numeric values can be assigned to the scaling factor, where values less than unity refer to size reduction and values greater than unity mean size enlargement. Central moments after a uniform scaling are defined as

$$\mu'_{pq} = \alpha^{p+q+2} \mu_{pq}. \tag{13}$$

The scale-normalized moments are

$$\mu'_{pq} = \frac{\mu_{pq}}{\mu_{00}^{\lambda}}, \quad \lambda = \frac{p+q+2}{2} \tag{14}$$

### 3.3. Normalization to first rotation

Rotation through an angle $\theta$ about the coordinate origin is represented by the following matrix form:

$$\mu_{pq}^{\mathrm{rot}} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x\cos\theta - y\sin\theta)^p (y\cos\theta + x\sin\theta)^q f(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{15}$$

By using the binomial theorem with Eq. (15), moments of the normalized image with respect to first rotation could be written as follows:

$$\mu_{pq}^{\mathrm{rot}} = \sum_{k=0}^{p} \sum_{j=0}^{q} \binom{p}{k} \binom{q}{j} (-1)^k (\sin\theta)^{q+k-j} (\cos\theta)^{p+j-k} \mu'_{p+q-k-j,k+j}. \tag{16}$$

Rotation normalization can be achieved by the major principal axis method [10]. The principal axis moments are obtained by rotating the axis of the central moments until $\mu_{11}^{rot}$ is zero. The angle $\theta$ is

$$\theta = -\frac{1}{2} \tan^{-1} \left( \frac{2\mu_{11}}{\mu_{20} - \mu_{02}} \right). \tag{17}$$

This method gives accurate results only in case of non-symmetric images and shapes, while it fails with the $N$-fold symmetrical objects with $N > 2$. This is a weak point, where many images and objects are symmetrical objects with fold greater than 2. The alternative approach is based on using the complex moment, where both original and rotating images have the same magnitude values of these moments, while the phase is shifted with the rotation angle as in the following:

$$c'_{pq} = \mathrm{e}^{\mathrm{i}(p-q)}\theta c_{pq}. \tag{18}$$

The idea to use the complex moments for deriving invariants was described first by Abu-Mostafa and Psaltis [11]. Complex moments of order $(p+q)$ for image intensity function $f(x,y)$ are defined as

$$c_{pq} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x + \mathrm{i}y)^p (x - \mathrm{i}y)^q f(x, y)\, \mathrm{d}x\, \mathrm{d}y, \tag{19}$$

where $\mathrm{i} = \sqrt{-1}$. By using the binomial theorem, each complex moment can be expressed as a combination of geometric moments of the same order or less as follows:

$$c_{pq} = \sum_{k=0}^{p} \sum_{j=0}^{q} \binom{p}{k} \binom{q}{j} (-1)^j i^{k+j} m_{p+q-k-j,k+j}. \tag{20}$$

Rotation normalization via complex moments requires $c'_{pq}$ being real and positive. $Ic_{pq}$ and $Rc_{pq}$ are the imaginary and real parts of the complex moment $c_{pq}$, and the rotation angle $\theta$ is evaluated as follows:

$$\theta = -\frac{1}{p-q} \tan^{-1}\left(\frac{Ic_{pq}}{Rc_{pq}}\right). \tag{21}$$

Any non-zero complex moment could be used for the rotation normalization process. It is preferable to keep the moment order as low as possible.

### 3.4. Normalization to stretching

Normalization to stretching can be done by imposing an additional constraint on the second-order moment [8]. This constraint is $\mu'_{20} = \mu'_{02}$, the stretching factor is defined as

$$\delta = \sqrt{\frac{\mu_{20} + \mu_{02} - \sqrt{(\mu_{20} - \mu_{02})^2 + 4\mu_{11}^2}}{2\sqrt{\mu_{20}\mu_{02} - \mu_{11}^2}}}. \tag{22}$$

Moments of the normalized image to stretching are

$$\mu''_{pq} = \delta^{p-q} \sum_{k=0}^{p} \sum_{j=0}^{q} \binom{p}{k} \binom{q}{j} (-1)^k (\sin\theta)^{q+k-j} (\cos\theta)^{p+j-k} \mu'_{p+q-k-j,k+j}. \tag{23}$$

### 3.5. Normalization to second rotation

For symmetric objects, many moments are to equal zero. According to the inaccurate approximation of moment values, zero-value moments may have a non-zero value. To deal with the symmetric images and objects, Suk and Flusser [8] used selected complex moments in the normalization process, where they assumed that the non-zero moments mean that their magnitude exceeds some threshold and the selection of this threshold is based on numerical experiment. This is a big challenge and ensures the requirement to accurate computation of moment invariants.

Complex moments are calculated using Eq. (20) where the geometric moments in the right side are replaced by $\mu''_{pq}$ from Eq. (23). The following algorithm is proposed for automatic selection of the normalization non-zero complex moment, where *Max* is the maximum order of the moment.

**for**$K = 3{:}Max$
  **for p**$= \lfloor \frac{K}{2} \rfloor + 1{:}\ K$;
    **q** $= Max - p$;
    **if**$(c_{pq} \neq 0)$
    {
      $Ic_{pq}$ = **imaginary**$(c_{p\ q})$;
      $Rc_{p\ q}$= **real** $(c_{p\ q})$ ;
      **Stop**
    }
  **endfor**
**endfor**

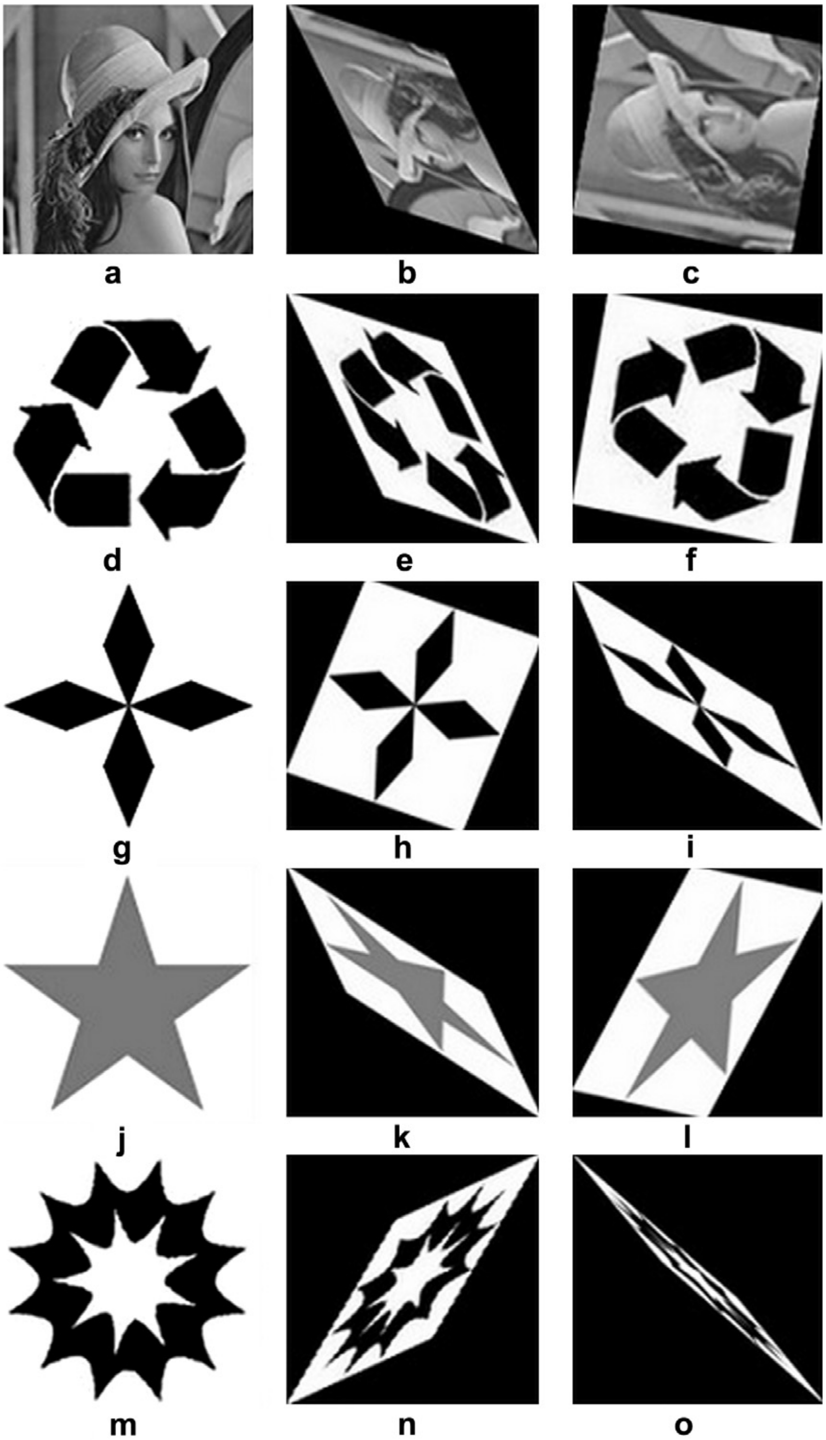

Fig. 1. (a) Original non-symmetric image of Lena; (b) and (c) are the transformed images. (d) Original symmetric image of recycle logo; (e) and (f) are the transformed images. (g) Original cross-shaped symmetric image; (h) and (i) are the transformed images. (j) Original five-point star symmetric image; (k) and (l) are the transformed images. (m) Original flower-shaped symmetric image; (n) and (o) are the transformed images.

After the selection of the first non-zero complex moment, the second rotation angle $\phi$ is

$$\phi = -\frac{1}{p-q}\tan^{-1}\left(\frac{Ic_{pq}}{Rc_{pq}}\right). \tag{24}$$

The affine moment invariants are

$$v_{pq} = \sum_{k=0}^{p}\sum_{j=0}^{q}\binom{p}{k}\binom{q}{j}(-1)^k(\sin\phi)^{q+k-j}(\cos\phi)^{p+j-k}\mu''_{p+q-k-j,k+j}. \tag{25}$$

Table 1
Invariance error using the proposed method

| | Transformed image (1) | Transformed image (2) | Invariance error |
|---|---|---|---|
| $v_{20}$ | 0.0404 | 0.0382 | 0.0022 |
| $v_{30}$ | −0.0003 | 0.0006 | 0.0009 |
| $v_{21}$ | 0.0003 | 0.0011 | 0.0008 |
| $v_{12}$ | −0.0009 | −0.0002 | 0.0007 |
| $v_{40}$ | 0.0039 | 0.0033 | 0.0006 |
| $v_{31}$ | −0.0001 | −0.0002 | 0.0003 |
| $v_{22}$ | 0.0008 | 0.0009 | 0.0001 |
| $v_{13}$ | 0.0001 | 0.0003 | 0.0002 |
| $v_{04}$ | 0.0037 | 0.0032 | 0.0005 |

Table 2
Invariance error using the ZOA method

| | Transformed image (1) | Transformed image (2) | Invariance error |
|---|---|---|---|
| $v_{20}$ | 0.1616 | 0.1530 | 0.0086 |
| $v_{30}$ | −0.0021 | 0.0050 | 0.0071 |
| $v_{21}$ | 0.0025 | 0.0091 | 0.0066 |
| $v_{12}$ | −0.0074 | −0.0052 | 0.0022 |
| $v_{40}$ | 0.0627 | 0.0519 | 0.0108 |
| $v_{31}$ | −0.0020 | −0.0063 | 0.0043 |
| $v_{22}$ | 0.0120 | 0.0152 | 0.0032 |
| $v_{13}$ | 0.0020 | 0.0044 | 0.0024 |
| $v_{04}$ | 0.0599 | 0.0474 | 0.0125 |

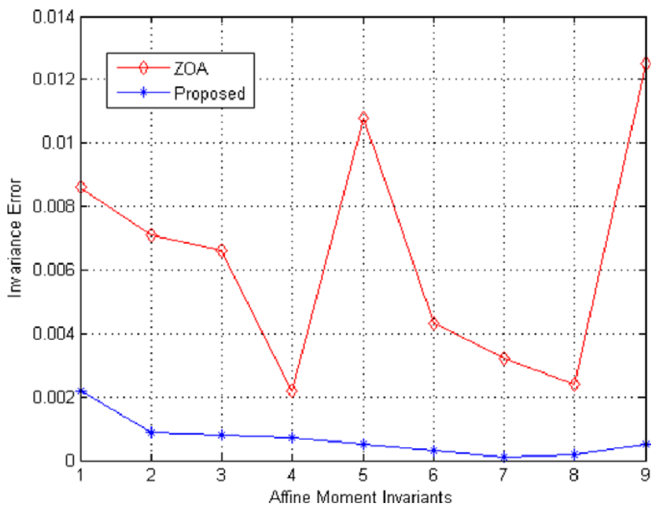


Fig. 2. Invariance error for the non-symmetric image of Lena.

Based on the normalization processes, the following affine moment invariants have specific values and are independent of the image or the object:

$$v_{00} = 1, \quad v_{10} = v_{01} = 0, \quad v_{20} = v_{02}, \quad v_{11} = 0, \quad v_{21} = -v_{03}. \tag{26}$$

## 4. Numerical experimental

In this section, numerical experiments with different non-symmetric and symmetric images are performed. In the first experiment, the standard gray-scale image of Lena of size $128 \times 128$ is used. This image is a non-symmetric image. The original image of Lena as in Fig. 1a is transformed by two different sets of affine transformations to produce the transformed images 1b and c.

Affine moment invariants of the second-, third- and fourth-order are used to test the invariance. According to Eq. (26), the rest of the selected affine moment invariants are

Second-order:

$$v_{20},$$

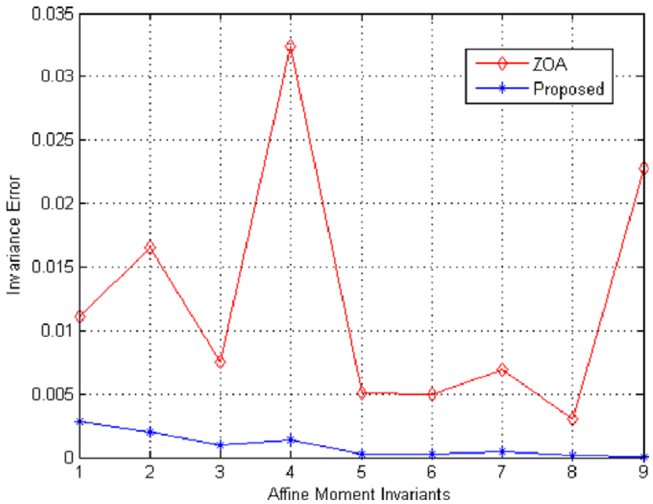


Fig. 3. Invariance error for the symmetric image of recycle logo.

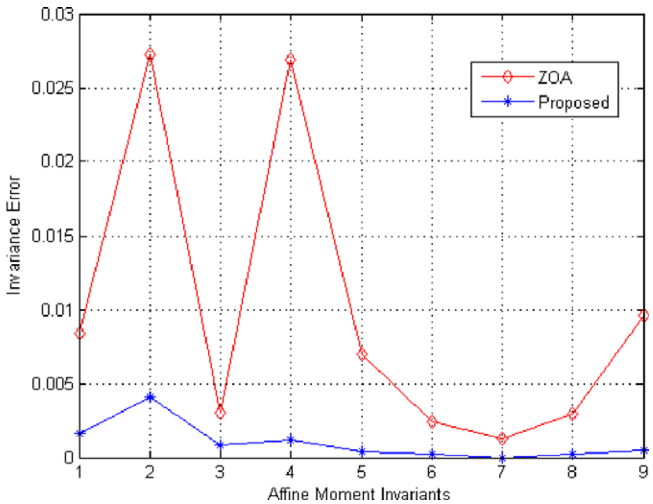


Fig. 4. Invariance error for the cross-shaped symmetric image.

Third-order:

$$v_{30}, \quad v_{21}, \quad v_{12}. \tag{27}$$

Fourth-order:

$$v_{40}, \quad v_{31}, \quad v_{22}, \quad v_{13}, \quad v_{04}.$$

The set of affine moment invariants in (27) are calculated for the original and transformed images. The calculated values using the proposed method are listed in Table 1, while those values of the ZOA method are listed in Table 2. Invariance errors are plotted in Fig. 2. It is clear that the invariance error of the proposed method is much smaller than ZOA values. The proposed method is stable and errors tend to zero as the order of the moment invariant increases. On the other side, the ZOA method is unstable where the errors fluctuate and tends to increase as the order of the moment invariant increases.

An image is said to be $N$-fold rotation symmetry with $N \geqslant 1$ if it repeats itself when it rotates around its centroid by $2\pi j/N$ for all $j = 1, 2, \ldots, N$. Based on this definition, gray-scale symmetric images with different

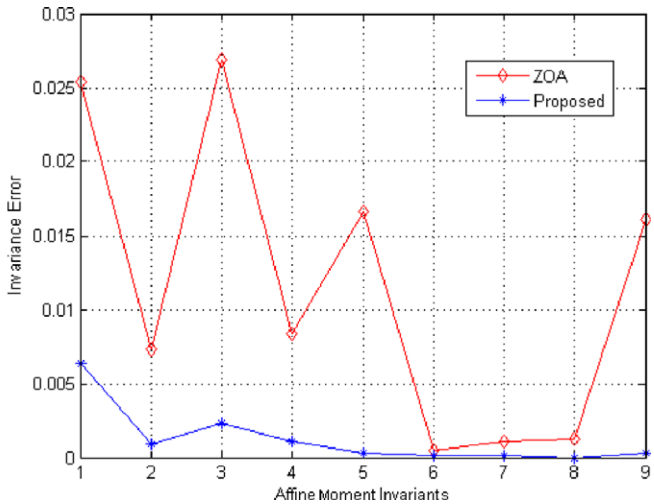


Fig. 5. Invariance error for the five-point star symmetric image.

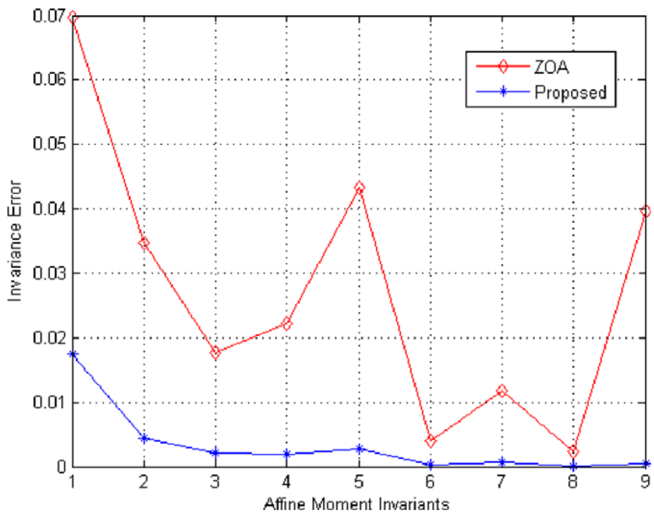


Fig. 6. Invariance error for the flower-shaped symmetric image.

$N$-fold rotation symmetry of size $128 \times 128$ are considered and used through the next numerical experiments. Symmetric images of recycle logo, cross-shaped, five-point star and flower-shaped are used in the second numerical experiment. As in the first experiment, all the original images are transformed using different groups of affine transformations. The invariance errors are plotted in Figs. 2–6. Similar to the case of non-symmetric image, the invariance error of the proposed method is much smaller than the corresponding one of the ZOA method. The plotted curves ensure the stability of the proposed method against the high fluctuations produced by the ZOA method.

## 5. Conclusion

This work proposes an accurate method to compute affine moment invariants for gray-scale images and objects. This problem is an important problem of pattern recognition applications. The calculation of the affine moment invariants for symmetric images usually leads to big ambiguities where many of these values are equal to zero. Accurate computation and automatic selection of normalization parameters overcome this problem. The numerical experiments are performed with symmetric as well as asymmetric images which ensure the stability and the accuracy of the proposed method.

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