$(2, 3, t)$-Generations for the Tits simple group ${}^2F_4(2)'$

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Abstract: A group G is said to be $(2, 3, t)$-generated if it can be generated by two elements x and y such that $o(x) = 2$, $o(y) = 3$ and $o(xy) = t$. In this paper, we determine $(2, 3, t)$-generations of the Tits simple group $\mathrm{T} \cong {}^2F_4(2)'$ where t is divisor of $|\mathrm{T}|$. Most of the computations were carried out with the aid of computer algebra system $\mathbb{GAP}$ [17].

Key–Words: Tits group ${}^2F_4(2)'$, simple group, $(2, 3, t)$-generation, generator.

1 Introduction

A group G is called $(2, 3, t)$-generated if it can be generated by an involution x and an element y of order 3 such that $o(xy) = t$. The $(2, 3)$-generation problem has attracted a vide attention of group theorists. One reason is that $(2, 3)$-generated groups are homomorphic images of the modular group $PSL(2, \mathbb{Z})$, which is the free product of two cyclic groups of order two and three. The motivation of $(2, 3)$-generation of simple groups also came from the calculation of the genus of finite simple groups [22]. The problem of finding the genus of finite simple group can be reduced to one of generations (see [24] for details).

Moori in [15] determined the $(2, 3, p)$-generations of the smallest Fischer group F_{22}. In [11], Ganief and Moori established $(2, 3, t)$-generations of the third Janko group J_3. In a series of papers [1], [2], [3], [4], [5], [12] and [13], the authors studied $(2, 3)$-generation and generation by conjugate elements of the sporadic simple groups Co_1, Co_2, Co_3, He, HN, Suz, Ru, HS, McL, Th and Fi_{23}. The present article is devoted to the study of $(2, 3, t)$-generations for the Tits simple group T, where t is any divisor of $|\mathrm{T}|$. For more information regarding the study of $(2, 3, t)$-generations, generation by conjugate elements as well as computational techniques used in this article, the reader is referred to [1], [2], [3], [4], [5], [11], [15], [16] and [22].

The Tits group $\mathrm{T} \cong {}^2F_4(2)'$ is a simple group of order $17971200 = 2^{11}.3^3.5^2.13$. The group T is a subgroup of the Rudvalis sporadic simple group Ru of index 8120. The group T also sits maximally inside the smallest Fischer group Fi_{22} with index 3592512. The maximal subgroups of the Tits simple group T was first determined by Tchakerian [19]. Later but independently, Wilson [20] also determined the maximal subgroups of the simple group T, while studying the geometry of the simple groups of Tits and Rudvalis.

For basic properties of the Tits group T and information on its subgroups the reader is referred to [20], [19]. The $\mathbb{ATLAS}$ of Finite Groups [9] is an important reference and we adopt its notation for subgroups, conjugacy classes, etc. Computations were carried out with the aid of $\mathbb{GAP}$ [17].

2 Preliminary Results

Throughout this paper our notation is standard and taken mainly from [1], [2], [3], [4], [5], [15] and [11]. In particular, for a finite group G with $C_1, C_2, \ldots, C_k$ conjugacy classes of its elements and g_k a fixed representative of C_k, we denote $\Delta(G) = \Delta_G(C_1, C_2, \ldots, C_k)$ the number of distinct tuples $(g_1, g_2, \ldots, g_{k-1})$ with $g_i \in C_i$ such that $g_1 g_2 \ldots g_{k-1} = g_k$. It is well known that $\Delta_G(C_1, C_2, \ldots, C_k)$ is structure constant for the conjugacy classes $C_1, C_2, \ldots, C_k$ and can easily be computed from the character table of G (see [14], p.45) by the following formula $\Delta_G(C_1, C_2, \ldots, C_k) = \frac{|C_1||C_2|\ldots|C_{k-1}|}{|G|} \times \sum_{i=1}^{m} \frac{\chi_i(g_1)\chi_i(g_2)\ldots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{[\chi_i(1_G)]^{k-2}}$ where $\chi_1, \chi_2, \ldots, \chi_m$ are the irreducible complex characters of G. Further, let $\Delta^*(G) = \Delta^*_G(C_1, C_2, \ldots, C_k)$ denote the number of distinct tuples $(g_1, g_2, \ldots, g_{k-1})$ with $g_i \in C_i$ and $g_1 g_2 \ldots g_{k-1} = g_k$ such that $G =< g_1, g_2, \ldots, g_{k-1} >$. If $\Delta^*_G(C_1, C_2, \ldots, C_k) > 0$, then we say that G is $(C_1, C_2, \ldots, C_k)$-generated.

If H is any subgroup of G containing the fixed element $g_k \in C_k$, then $\Sigma_H(C_1, C_2, \ldots, C_{k-1}, C_k)$ denotes the number of distinct tuples $(g_1, g_2, \ldots, g_{k-1}) \in (C_1 \times C_2 \times \ldots \times C_{k-1})$ such that $g_1 g_2 \ldots g_{k-1} = g_k$ and $\langle g_1, g_2, \ldots, g_{k-1} \rangle \leq H$ where $\Sigma_H(C_1, C_2, \ldots, C_k)$ is obtained by summing the structure constants $\Delta_H(c_1, c_2, \ldots, c_k)$ of H over all H-conjugacy classes $c_1, c_2, \ldots, c_{k-1}$ satisfying $c_i \subseteq H \cap C_i$ for $1 \leq i \leq k-1$.

For the description of the conjugacy classes, the character tables, permutation characters and information on the maximal subgroups readers are referred to $\mathbb{ATLAS}$ [9]. A general conjugacy class of elements of order n in G is denoted by nX. For example $2A$ represents the first conjugacy class of involutions in a group G.

The following results in certain situations are very effective at establishing non-generations.

Theorem 1 *(Scott's Theorem, [8] and [18]) Let $x_1, x_2, \ldots, x_m$ be elements generating a group G with $x_1 x_2 \cdots x_n = 1_G$, and V be an irreducible module for G of dimension $n \geq 2$. Let $C_V(x_i)$ denote the fixed point space of $\langle x_i \rangle$ on V, and let d_i is the codimension of $V/C_V(x_i)$. Then $d_1 + d_2 + \cdots + d_m \geq 2n$.*

Lemma 2 *([8]) Let G be a finite centerless group and suppose lX, mY, nZ are G-conjugacy classes for which $\Delta^*(G) = \Delta^*_G(lX, mY, nZ) < |C_G(z)|, z \in nZ$. Then $\Delta^*(G) = 0$ and therefore G is not (lX, mY, nZ)-generated.*

3 $(2, 3, t)$-Generations of Tits group

The Tits group $\mathrm{T} \cong {}^2F_4(2)'$ has 8 conjugacy classes of its maximal subgroups as determined by Wilson [20] and listed in the $\mathbb{ATLAS}$ [9]. The group T has 22 conjugacy classes of its elements including 2 involutions namely $2A$ and $2B$.

In this section we investigate $(2, 3, t)$-generations for the Tits group T where t is a divisor of $|\mathrm{T}|$. It is a well known fact that if a group G is $(2, 3, t)$-generated simple group, then $1/2 + 1/3 + 1/t < 1$ (see [7] for details). It follows that for the $(2, 3, t)$-generations of the Tits simple group T, we only need to consider $t \in \{8, 10, 12, 13, 16\}$.

Lemma 3 *The Tits simple group T is not $(2A, 3A, tX)$-generated for any $tX \in \{8A, 8B, 8C, 8D, 10A\}$.*

Proof. For the triples $(2A, 3A, 8A)$ and $(2A, 3A, 8B)$ non-generation follows immediately since the structure constants $\Delta_{\mathrm{T}}(2A, 3A, 8A) = 0$ and $\Delta_{\mathrm{T}}(2A, 3A, 8B) = 0$.

The group T acts on 78-dimensional irreducible complex module V. We apply Scott's theorem (cf. Theorem 1) to the module V and compute that

$$\begin{aligned} d_{2A} &= dim(V/C_V(2A)) = 32, \\ d_{3A} &= dim(C/C_V(3A)) = 54 \\ d_{8C} &= dim(V/C_V(8C)) = 68, \\ d_{8D} &= dim(V/C_V(8D)) = 68 \\ d_{10A} &= dim(V/C_V(10A)) = 68 \end{aligned}$$

Now, if the group T is $(2A, 3A, tX)$-generated, where $tX \in \{8C, 8D, 10A\}$, then by Scott's theorem we must have

$$d_{2A} + d_{3A} + d_{tX} \geq 2 \times 78 = 156.$$

However, $d_{2A} + d_{3A} + d_{tX} = 154$, and non-generation of the group T by these triples follows. $\square$

Lemma 4 *The Tits simple group T is $(2B, 3A, 8Z)-$generated, where $Z \in \{A, B, C, D\}$ if and only if $Z = A$ or B.*

Proof. Our main proof will consider the following three cases.

Case $(2B, 3A, 8Z)$*, where* $Z \in \{A, B\}$: We compute $\Delta_T(2B, 3A, 8Z) = 128$. Amongst the maximal subgroup of T, the only maximal subgroups having non-empty intersection with any conjugacy class in the triple $(2B, 3A, tZ)$ is isomorphic to $H \cong 2^2.[2^8]{:}S_3$. However $\Sigma_H(2B, 3A, 8Z) = 0$, which means that H is not $(2B, 3A, 8Z)$-generated. Thus $\Delta^*_{\mathrm{T}}(2B, 3A, 8Z) = \Delta_T(2B, 3A, 8Z) = 128 > 0$, and the $(2B, 3A, 8Z)$-generation of T, for $Z \in \{A, B\}$, follows.

Case $(2B, 3A, 8C)$: The only maximal subgroups of the group T that may contain $(2B, 3A, 8C)$-generated subgroups, up to isomorphism, are $H_1 \cong L_3(3){:}2$ (two non-conjugate copies) and $H_2 \cong 2^2.[2^8]{:}S_3$. Further, a fixed element $z \in 8C$ is contained in two conjugate subgroup of each copy of H_1 and in a unique conjugate subgroup of H_2. A simple computation using $\mathbb{GAP}$ reveals that $\Delta_T(2B, 3A, 8C) = 112$, $\Sigma_{H_1}(2B, 3A, 8C) = \Sigma_{L_3(3)}(2B, 3A, 8C) = 20$ and $\Sigma_{H_2}(2B, 3A, 8C) = 32$. By considering the maximal subgroups of $H_{11} \cong L_3(3)$ and H_2, we see that no maximal subgroup of H_{11} and H_2 is (2B,3A,8C)-generated and hence no proper subgroup of H_{11} and H_2 is $(2B, 3A, 8C)$-generated. Thus,

$$\begin{aligned} \Delta^*_{\mathrm{T}}(2B, 3A, 8C) &= \Delta_{\mathrm{T}}(2B, 3A, 8C) \\ &\quad -4\Sigma^*_{H_{11}}(2B, 3A, 8S) \\ &\quad -\Sigma^*_{H_2}(2B, 3A, 8C) \\ &= 112 - 4(20) - 32 = 0. \end{aligned}$$

Therefore, the Tits simple group T is not $(2B, 3A, 8C)$-generated.

Case $(2B, 3A, 8D)$: In this case, $\Delta_{\rm T}(2B, 3A, 8D) = 112$. We prove that Tits simple group T is not $(2B, 3A, 8D)$-generated by constructing the $(2B, 3A, 8D)$-generated subgroup of the group He explicitly. We use the "standard generators" of the group T given by Wilson in [21]. The group T has a 26-dimensional irreducible representation over $\mathbb{GF}(2)$. Using this representation we generate the Tits group $T = \langle a, b\rangle$, where a and b are 26×26 matrices over $\mathbb{GF}(2)$ with orders 2 and 3 respectively such that ab has order 13. Using $\mathbb{GAP}$, we see that $a \in 2A$, $b \in 3A$. We produce $c = (ababab^2)^6$, $p = abababab^2abab^2ab^2$, $d = (acp)^6$, $x = p^{16}dp^{-16}$ such that $c, d, x \in 2B$, $p \in 10A$ and $xb \in 8D$. Let $H = \langle x, b\rangle$ then $H < {\rm T}$ with $H \cong L_3(3){:}2$. Since no maximal subgroup of H is $(2B, 3A, 8D)$-generated, that is no proper subgroup of H is $(2B, 3A, 8D)$-generated and we have $\Sigma^*_H(2B, 3A, 8D) = \Sigma_H(2B, 3A, 8D)$. Since $\Sigma_H(2B, 3A, 8D) = 28$ and $z \in 8D$ is contained in exactly two conjugate subgroups of each copy of H, we obtain that $\Delta^*_{\rm T}(2B, 3A, 8D) = 0$. Hence the Tits simple group T is not $(2B, 3A, 8D)$-generated. This completes the lemma. □

Lemma 5 *The Tits group T is* $(2B, 3A, 10A)$*−generated.*

Proof. Up to isomorphism, the only maximal subgroups having non-empty intersection with any conjugacy class in the triple $(2B, 3A, 10A)$ are isomorphic to $H \cong 2^2.[2^8]{:}S_3$, $K \cong A_6{\cdot}2^2$(two non-conjugate copies). Since $\Delta_T(2B, 3A, 10A) = 100$ and $\Sigma_H(2B, 3A, 10A) = 0 = \Sigma_K(2B, 3A, 10A)$. we conclude that no maximal subgroup of T is $(2B, 3A, 10A)$-generated. Thus

$$\Delta^*_{\rm T}(2B, 3A, 10A) = \Delta_{\rm T}(2B, 3A, 10A) = 100$$

and the $(2B, 3A, 10A)$-generation of Tits group T follows. □

Lemma 6 *The Tits group T is not* $(2X, 3A, 12Z)$*−generated where* $X, Z \in \{A, B\}$.

Proof. First we consider the case $X = A$. The maximal subgroups of the group T that may contain $(2A, 3A, 12Z)$-generated subgroups are isomorphic to $H \cong 2^2.[2^8]{:}S_3$ and $K \cong 5^2{:}4A_4$ (two non-conjugate copies). We compute that $\Delta_T(2A, 3A, 12Z) = 32$, $\Sigma_H(2A, 3A, 12Z) = 12$ and $\Sigma_K(2A, 3A, 12Z) = 15$. A fixed element of order 12 in T is contained in a unique conjugate subgroup of H and two conjugate subgroups of K. Since no maximal subgroup of each H and K is $(2A, 3A, 12Z)$-generated, we obtain

$$\begin{aligned}\Delta^*_{\rm T}(2A, 3A, 12Z) &= \Delta_{\rm T}(2A, 3A, 12Z)\\ &\quad -\Sigma^*_H(2A, 3B, 12Z)\\ &\quad -4\Sigma^*_K(2A, 3A, 12Z)\\ &= 32 - 12 - 2(15) < 0\end{aligned}$$

and the non-generation of the group Tits by the triple $(2A, 3A, 12Z)$ follows.

Next, suppose That $X = B$. There are six maximal subgroups of the group T having non-empty intersection with each conjugacy class in the triple $(2B, 3A, 12Z)$, are isomorphic to $H_1 = L_3(3){:}2$ (two non-conjugate copies), $H_2 \cong L_2(25)$, $H_3 \cong 2^2.[2^8]{:}S_3$ and $H_4 = 5^2{:}4A_4$ (two non-conjugate copies). Further, a fixed element of order 12 in Tits group is contained in a unique conjugate subgroups of each of H_1, H_2, H_3 and H_4. We calculate $\Delta_{\rm T}(2B, 3A, 12Z) = 84$, $\Sigma_{H_1}(2B, 3A, 12Z) = 27$, $\Sigma_{H_2}(2B, 3A, 12Z) = 24$, $\Sigma_{H_3}(2B, 3A, 12Z) = 12$ and $\Sigma_{H_4}(2B, 3A, 12Z) = 0$. Since no maximal subgroup of each of the groups H_1, H_2, H_3 and H_4 is $(2B, 3A, 12Z)$-generated. We conclude that

$$\begin{aligned}\Delta^*_{\rm T}(2B, 3A, 12Z) &= \Delta_{\rm T}(2B, 3A, 12Z)\\ &\quad -2\Sigma^*_{H_1}(2B, 3A, 12Z)\\ &\quad -\Sigma^*_{H_2}(2B, 3A, 12Z)\\ &\quad -\Sigma^*_{H_3}(2B, 3A, 12Z)\\ &= 84 - 2(27) - 24 - 12 < 0.\end{aligned}$$

Therefore Tits group T is not $(2B, 3A, 12Z)$-generated. This completes the proof. □

Lemma 7 *The Tits group T is* $(2X, 3A, 13Z)$*-generated where* $X, Z \in \{A, B\}$ *if and only if* $X = A$

Proof. First we consider the case $X = A$. The structure constant $\Delta_{\rm T}(2A, 3A, 13Z) = 13$. The fusion maps of the maximal subgroup of Tits group T into the group T shows that there is no maximal subgroup of T has non-empty intersection with the classes in the triple $(2A, 3A, 13Z)$. That is no maximal subgroup of T is $(2A, 3A, 13Z)$-generated. Hence,

$$\Delta^*_{\rm T}(2A, 3A, 13Z) = \Delta_{\rm T}(2A, 3A, 13Z) = 13 > 0$$

which implies that the Tits group T is $(2A, 3A, 13Z)$-generated for $Z \in \{A, B\}$.

Next suppose that $X = B$. Up to isomorphism, the only maximal subgroups of T having non-empty intersection with each conjugacy class in the triple $(2B, 3A, 13Z)$ are isomorphic to $L_3(3){:}2$ (two non-conjugate copies) and $L_2(25)$.

Further a fixed element of order 13 in the Tits group T is contained in a unique conjugate of each of $L_3(3)$:2 and in three conjugate of $L_2(25)$ subgroups. We compute that $\Delta_T(2B, 3A, 13Z) = 104$, $\Sigma_{L_3(3):2}(2B, 3A, 13Z) = \Sigma_{L_3(3)}(2B, 3A, 13A) = 13$ and $\Sigma_{L_2(25)}(2B, 3A, 13Z) = 26$. Now by considering the maximal subgroups of $L_3(3)$ and $L_2(25)$, we see that no maximal subgroup of the groups $L_3(3)$ and $L_2(25)$ is $(2B, 2A, 13Z)$-generated. It follows that no proper subgroup of $L_3(3)$ or $L_2(25)$ is $(2B, 3A, 13Z)$-generated. Thus we have

$$\begin{aligned}\Delta^*_{\mathrm{T}}(2B, 3A, 13Z) &= \Delta_{\mathrm{T}}(2B, 3A, 13Z)\\ &\quad -2\Sigma^*_{L_3(3)}(2B, 3A, 13Z)\\ &\quad -3\Sigma^*_{L_2(25)}(2B, 3A, 13Z)\\ &= 104 - 2(13) - 3(26) - 12 = 0,\end{aligned}$$

proving non-generation of the Tits group T by the triple $(2B, 3A, 13Z)$, where $Z \in \{A, B\}$. □

Lemma 8 *The Tits group T is* $(2X, 3A, 16Z)$*-generated, where* $X \in \{A, B\}$ *and* $Z \in \{A, B, C, D\}$.

Proof. We treat two cases separately.

Case (2A,3A,16Z): The structure constant $\Delta_{\mathrm{T}}(2A, 3A, 16Z) = 16$. We observe that the group isomorphic to $2^2.[2^8]{:}S_3$ is the only maximal subgroup of T that may contain $(2A, 3A, 16Z)$-generated subgroups. However we calculate $\Sigma_H(2A, 3A, 16Z) = 0$ for $H \cong 2^2.[2^8]{:}S_3$ and hence $\Delta^*_T(2A, 3A, 16Z) = \Delta_T(2A, 3A, 16Z) = 16 > 0$, proving that $(2A, 3A, 16Z)$ is a generating triple of the Tits group.

Case (2B,3A,16Z): Up to isomorphism, $H \cong 2^2.[2^8]{:}S_3$ is the only one maximal subgroup of T that may admit $(2B, 3A, 16Z)$-generated subgroups. A fixed element of order 16 in the Tits group T is contained in a unique conjugate subgroups of H. Since $\Delta_T(2B, 3A, 16Z) = 112, \Sigma_H(2B, 3A, 16Z) = 32$, we conclude that

$$\Delta^*_{\mathrm{T}}(2B, 3A, 16Z) \geq 112 - 32 = 80 > 0$$

and the $(2B, 3A, 16Z)$-generation of T follows. □

We now summarize our results in the next theorem.

Theorem 9 *Let* tX *be a conjugacy class of the Tits simple group T. The group T is* $(2A, 3A, tX)$*-generated if and only if* $tX \in \{13Y, 16Z\}$ *where* $Y \in \{A, B\}$ *and* $Z \in \{A, B, C, D\}$*. Further, the group T is* $(2B, 3A, tX)$*-generated if and only if* $tX \in \{8Y, 10A, 16Z\}$.

Proof. This is merely a restatement of the lemmas in this section. □

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